$$\mathfrak{K}_1 = -\frac{\partial \mathcal{H}_1}{\partial \mathfrak{r}_1} = -\frac{e_1\, e_2\, (\mathfrak{r}_2 - \mathfrak{r}_1)}{4\,\pi\, r^3} \sqrt{1 - \frac{\mathfrak{v}_1^2}{c^2}}\,. \tag{20}$$

Damit hat sich unsere Vermutung bezüglich Formel (11) bestätigt.

Diese letzte Gleichung resultiert auch sofort aus transformationstheoretischen Gründen: Die Kraftdichte k_μ transformiert sich bei einer speziellen LORENTZ-Transformation wie ein Vierervektor, so daß

$$k_{1y}' = k_{1y}$$

gilt. Da sich das Volumelement wie

$$\mathrm{d}V_1' = \mathrm{d}V_1 \sqrt{1 - \mathfrak{v}_1^2/c^2}$$

transformiert, ergibt sich für die Kraft

$$K_{1y}' = K_{1y} \sqrt{1 - \mathfrak{v}_1^2/c^2}.$$

Identifizieren wir das ungestrichene Bezugssystem mit dem Ruhsystem der beiden Teilchen, in dem für K_{1y} die COULOMB-Kraft steht, so finden wir genau (20).

On the Rôle of the Group O_4 of Local Complex Orthogonal Transformations in a Nonlinear Theory of Elementary Particles

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It is supposed that there exists a system O' (intrinsic system) in which the field equation for a spin $\frac{1}{2}$ representation has the simple form $\gamma^\mu\, \partial\psi'/\partial x^{\mu'} = 0$. This system is related to the physical system (in which all measurements are performed) by an affine connection which is induced by a certain group of local transformations. The investigation given here deals with the group of local four-dimensional complex orthogonal transformations. Subjecting ψ' to such a transformation Ω one gets with $\psi'(x') = \Omega(x)\, \psi(x)$ the following equation $\gamma_\lambda\, \partial\psi/\partial x_\lambda + \gamma^\lambda\, \Omega^{-1}\, \partial\Omega/\partial x^\lambda\, \psi = 0$. The interaction term splits up into a vector and a pseudovector part: $\gamma^\lambda\, \Omega^{-1}\, \partial\Omega/\partial x^\lambda \equiv \gamma^\lambda\, V_\lambda + \gamma^\lambda\, \gamma^5\, P_\lambda$. The special cases of real local orthogonal (LORENTZ-) transformations ($\xi_{\lambda\mu} = -\xi_{\mu\lambda}$; ξ_{kl} real, ξ_{4l} imaginary; $\psi \to \chi$) and special complex local orthogonal transformations ($\eta_{\lambda\mu} = -\eta_{\mu\lambda}$; η_{kl} imaginary, η_{4l} real; $\psi \to \varphi$) are first separately considered. It is required that V_λ and P_λ are to be built up from the fundamental covariants of the field. In order that certain conservation laws hold at least approximately, the following assumptions are made:

$$\operatorname{Im}\{V_k\} = \pm k^2\, \bar\varphi\, \gamma_k\, \varphi\,, \quad \operatorname{Re}\{V_4\} = \pm k^2\, \bar\varphi\, \gamma_4\, \varphi\,, \quad \operatorname{Im}\{P_k\} = \pm l^2\, \bar\chi\, \gamma_k\, \gamma_5\, \chi\,, \quad \operatorname{Re}\{P_4\} = \pm l^2\, \bar\chi\, \gamma_4\, \gamma_5\, \chi$$

together with the symmetry conditions for the transformation parameters, $\xi_{\lambda[\mu,\nu]} \equiv 0$, $\eta_{\langle\lambda\mu,\nu\rangle} \equiv 0$, which can be fulfilled by setting, for example, $\xi_{\lambda\mu,\nu} = \pi_{[\lambda}\, \pi_{\mu,\nu]}$, $\eta_{\lambda\mu} = \vartheta_{[\lambda,\mu]}$. The remaining parts of V_λ and P_λ, which are determined by these relations, are of higher order and can be assumed to describe weaker interactions. Neglecting these terms one obtains the following set of equations:

$$\text{(a)} \qquad \gamma^\lambda\, \partial\chi/\partial x^\lambda \pm k^2\, \gamma^\lambda (\bar\varphi\, \gamma_\lambda\, \varphi)\, \chi \pm l^2\, \gamma^\lambda\, \gamma^5 (\bar\chi\, \gamma_\lambda\, \gamma_5\, \chi)\, \chi \approx 0\,,$$

$$\text{(b)} \qquad \gamma^\lambda\, \partial\varphi/\partial x^\lambda \pm k^2\, \gamma^\lambda (\bar\varphi\, \gamma_\lambda\, \varphi)\, \varphi \pm l^2\, \gamma^\lambda\, \gamma^5 (\bar\chi\, \gamma_\lambda\, \gamma_5\, \chi)\, \varphi \approx 0\,.$$

Since the pseudovector coupling possesses a greater symmetry, it is assumed that χ represents the baryon and φ the lepton states. Within the approximation, which holds with (a) and (b), it follows the conservation of $\bar\chi\, \gamma_\lambda\, \chi$ and $\bar\varphi\, \gamma_\lambda\, \varphi$ resp. (conservation of electric charge) and $\bar\chi\, \gamma_\lambda\, \gamma_5\, \chi$ and $\bar\varphi\, \gamma_\lambda\, \gamma_5\, \varphi$ resp. (conservation of baryonic and leptonic charge resp.). These conservation laws are exact only if the mentioned terms of higher order are neglected; this is equivalent to a strict "local" conservation as can be shown. As to the isospin it is proposed to replace one of its components by a bounded state, i. e. a mixture of χ- and φ-states which would lead in the case of the neutron for example to the components of the β-decay. Due to the relations $\pm k^2\, \bar\varphi\, \gamma^\lambda\, \varphi = \frac{1}{4}\, \eta^{\lambda\varrho}{}_{,\varrho} + O(\eta^2)$ and $\eta_{\lambda\mu} = \vartheta_{[\lambda,\mu]}$, and in agreement with the reality conditions, it is possible to connect the parameters ϑ_λ with the electromagnetic field A_λ by setting $\vartheta_\lambda = 8\, i\, A_\lambda$. Taking into consideration terms of higher order this would lead to a type of nonlinear electrodynamics.

In two papers [1, 2] recently published, it has been shown that the nonlinear term of the HEISENBERG–PAULI-equation can be interpreted as an affine connection which is induced by local LORENTZ-transformations. The heuristic principle which leads to this result can be outlined as follows: Suppose there

[1] V. I. RODICEV, Soviet Phys.—JETP **40**, 1169 [1961].

[2] G. BRAUNSS, Z. Naturforschg. **19 a**, 825 [1964].

exists always a system O' ("intrinsic" system) in which the field equations are as simple as possible, i. e. linear. This system is related to the "physical" system O (in which all measurements are performed) by an affine connection. This connection is induced by a certain group of local transformations. Changing from O' to O these transformations produce in the field equations additional terms which may be interpreted as describing interactions [3]. Since the corresponding transformations are restricted by certain conditions, such as reality conditions, one may expect only certain types of interaction. This procedure leads of course not to any simplifications as far as practical calculations are considered. But it permits a classification of interactions with respect to the corresponding affine connections resp. the groups which induce these connections. Furthermore, it may be of advantage to use this point of view for the quantization of the field: Since the field equations are linear with respect to O' one can write down immediately the commutator function in the "intrinsic" system and then pass over to the "physical" system by means of the corresponding local transformation which has of course to be determined as a solution of the nonlinear equation in O. (This reminds of the procedure one uses in the interaction picture and is apparently related to it.) However, we shall not deal here with this problem but treat the field throughout as a classical one. As to the affine connections we restrict ourselves to those which are induced by the group O_4 of (proper) local fourdimensional complex orthogonal transformations.

I.

In the case of a field with a spin $\frac{1}{2}$ representation, which we regard as fundamental for well known reasons, we assume the following equation to be the most simple one with respect to the "intrinsic" system:

$$\gamma^{\mu} \frac{\partial \psi'}{\partial x^{\mu'}} = 0 . \tag{1}$$

Since we shall permit not only real but in general complex orthogonal transformations, we must assume $\psi'(x')$ to be a four-component spinor by taking into consideration that we have the following homomorphism: $O_4 \sim C_2 \times C_2'$. Hence, ψ is reducible to a two-component spinor only if we consider real or special complex transformations. One might object that a complex orthogonal transformation leaves the norm invariant yet "mixes up" the real and imaginary components and has therefore no sense. To answer this we point to the fact that such a mixing happens only in the "intrinsic" system, whereas all measurements have to be performed in the "physical" system which has by condition the right reality properties (x^1, x^2, x^3 real, x^4 imaginary). This will become clearer from the following.

Let $\Omega = \Omega(x)$ be a complex orthogonal transformation. Taking a representation in the Dirac ring we can define such a transformation by

$$\Omega = \exp\{\tfrac{1}{4}\, \omega_{\varrho\sigma}\, \gamma^{\varrho}\, \gamma^{\sigma}\} , \quad \omega_{\varrho\sigma} = -\, \omega_{\sigma\varrho} . \tag{2}$$

With

$$\omega_{\varrho\sigma} = \xi_{\varrho\sigma} + \eta_{\varrho\sigma} , \quad \xi_{\varrho\sigma} = -\, \xi_{\sigma\varrho} , \; \eta_{\varrho\sigma} = -\, \eta_{\sigma\varrho} \tag{3}$$

and the reality conditions

$$\xi_{kl} , \eta_{4l} \text{ real; } \xi_{4l} , \eta_{kl} \text{ imaginary } (k, l = 1, 2, 3) \tag{4}$$

we may write

$$L = L(\xi(x)) , \quad \Lambda = \Lambda(\eta(x)) \tag{5}$$

where L denotes a real transformation (Lorentz-transformation) and Λ a transformation which we will call a special complex transformation. By writing down an infinitesimal transformation we may prove the following relations ($(c_{ik})^+ = (c^*_{ki})$, $(c_{ik})^{\mathrm{T}} = (c_{ki})$):

$$\bar{L} = \gamma_4\, L^+\, \gamma_4 = L^{-1}, \quad \bar{\Lambda} = \gamma_4\, \Lambda^+\, \gamma_4 = \Lambda . \tag{6 a, b}$$

Introducing Schwinger's matrix which is defined by

$$C\, \gamma^{\mu}\, C^{-1} = -\, \gamma^{\mu \mathrm{T}}, \quad C\, \gamma^5\, C^{-1} = \gamma^{5\mathrm{T}}, \tag{7 a, b}$$

we obtain further

$$C^{-1}\, L^{\mathrm{T}}\, C = L^{-1}, \; C^{-1}\, \Lambda^{\mathrm{T}}\, C = \Lambda^{-1}, \; C^{-1}\, \Omega^{\mathrm{T}}\, C = \Omega^{-1}. \tag{8 a, b, c}$$

Since Ω is an orthogonal transformation we have

$$\Omega\, \gamma^{\lambda}\, \Omega^{-1} = a^{\lambda}{}_{\varrho}\, \gamma^{\varrho}, \quad a_{\lambda\varrho}\, a_{\mu}{}^{\varrho} = \delta_{\lambda\mu} . \tag{9}$$

Now, if we subject $\psi'(x')$ to an orthogonal transformation by

$$\psi'(x') = \Omega(x)\, \psi(x) \tag{10}$$

we get from (1) with (9) and $a^{\lambda}{}_{\mu} = \partial x^{\lambda} / \partial x^{\mu'}$

$$0 = \Omega\, \gamma^{\mu} \frac{\partial \psi'}{\partial x^{\mu'}} = \gamma^{\lambda} \frac{\partial \psi}{\partial x^{\lambda}} + \gamma^{\lambda}\, \Omega^{-1} \frac{\partial \Omega}{\partial x^{\lambda}}\, \psi . \tag{11}$$

[3] The introduction of additional terms in the Dirac- and Klein–Gordon-equation on the basis of local transformations has been studied by several authors; see for example A. M. Brodskij et al., Soviet Phys.—JETP 14, 930 [1962].

Using the following formula for the derivative of an operator exponential [4],

$$\frac{\mathrm{d}}{\mathrm{d}y} e^{A(y)} = e^{A(y)} \int_0^1 e^{-\zeta A(y)} \frac{\mathrm{d}A}{\mathrm{d}y} e^{\zeta A(y)} \,\mathrm{d}\zeta\,, \qquad (12)$$

we get with (2) and (9) (comma denotes partial differentiation)

$$\gamma^\lambda \Omega^{-1} \frac{\partial \Omega}{\partial x^\lambda} = \frac{1}{4} \gamma^\lambda \gamma^\alpha \gamma^\beta \omega_{\varrho\sigma,\lambda} \int_0^1 a^\varrho{}_\alpha(\zeta)\, a^\sigma{}_\beta(\zeta)\,\mathrm{d}\zeta \quad (13)$$

where

$$e^{-\zeta\omega} \gamma^\lambda e^{\zeta\omega} = a^\lambda{}_\mu(\zeta)\,\gamma^\mu, \quad \omega = \tfrac{1}{4}\,\omega_{\varrho\sigma}\gamma^\varrho\gamma^\sigma. \quad (14)$$

With the abbreviations

$$C_{\alpha\beta\lambda} = -C_{\beta\alpha\lambda} = \tfrac{1}{4}\,\omega_{\varrho\sigma,\lambda} \int_0^1 a^\varrho{}_\alpha(\zeta)\, a^\sigma{}_\beta(\zeta)\,\mathrm{d}\zeta \qquad (15)$$

and

$$C_{\langle\varrho\sigma\tau\rangle} \equiv C_{\varrho\sigma\tau} + C_{\sigma\tau\varrho} + C_{\tau\varrho\sigma}\,, \quad (\varrho \neq \sigma \neq \tau \neq \varrho) \quad (16)$$

we obtain finally

$$\gamma^\lambda \Omega^{-1} \frac{\partial\Omega}{\partial x^\lambda} = \gamma^\lambda C_{\lambda\varrho}{}^\varrho + \gamma_\lambda\gamma_5(-1)^\lambda C_{\langle\varrho\sigma\tau\rangle}\ (\lambda \neq \varrho,\sigma,\tau). \quad (17)$$

It is obvious that

$$V_\lambda \equiv C_{\lambda\varrho}{}^\varrho \text{ resp.} \quad P_\lambda \equiv (-1)^\lambda C_{\langle\varrho\sigma\tau\rangle}\ (\lambda \neq \varrho,\sigma,\tau) \quad (18\,\mathrm{a,b})$$

are the components of a vector and a pseudovector respectively. We may further conclude that $C_{\lambda\mu\nu} = -C_{\mu\lambda\nu}$ are the components of an affine connection [5].

Going back to (13) we see that all the following equations remain unaltered if we replace Ω by $\Omega_0\,\Omega$ where Ω_0 is a constant nonsingular matrix. Hence, we may choose our initial condition in a way that at a fixed but arbitrary point $x = x_0$ we have $\Omega(x_0) = 1$ which gives with

$$C_\lambda{}^\mu{}_\nu = \omega_{\lambda\mu,\nu}(x_0) + \omega_{\lambda\mu,\nu\varrho}(x_0)\,(x^\varrho - x_0{}^\varrho) + \dots$$

the identities

$$V^\varrho{}_{,\varrho}(x_0) \equiv 0\,, \quad P^\varrho{}_{,\varrho}(x_0) \equiv 0\,. \quad (19\,\mathrm{a,b})$$

II.

If we restrict ourselves to real orthogonal transformations with $\omega_{\lambda\mu} = \xi_{\lambda\mu}$, we have $O_4 = N_4 \sim C_2$ and the representation becomes reducible to a two-component one. Yet for the sake of formal simplicity we shall use the four-component notation. We denote the spinor here by χ and mark the vector and pseudovector by an upper index: $V_\lambda{}^{(L)} \equiv V_\lambda(L)$, $P_\lambda{}^{(L)} \equiv P_\lambda(L)$. The reality properties can be obtained from (17). We find

$$V_k{}^{(L)},\ P_4{}^{(L)} \text{ real};\ V_4{}^{(L)}, P_k{}^{(L)} \text{ imaginary} \qquad (20)$$
$$(k = 1, 2, 3).$$

The question we shall ask now is: How to identify $V_\lambda{}^{(L)}$ and $P_\lambda{}^{(L)}$? Since we consider in the sense of HEISENBERG the spinor field as fundamental, $V_\lambda{}^{(L)}$ and $P_\lambda{}^{(L)}$ must be covariants of the field. Taking into consideration the reality conditions this would suggest $V_\lambda{}^{(L)} \sim i\,\bar\chi\,\gamma_\lambda\,\chi$ and $P_\lambda{}^{(L)} \sim \bar\chi\,\gamma_\lambda\,\gamma_5\,\chi$. But the first of these relations must be excluded for the following reason: If we write down the field equation we have

$$\gamma^\lambda \frac{\partial\chi}{\partial x^\lambda} + \gamma^\lambda V_\lambda{}^{(L)}\chi + \gamma^\lambda\gamma^5 P_\lambda{}^{(L)}\chi = 0\,. \qquad (21\,\mathrm{a})$$

Taking the adjoint we get

$$\frac{\partial\bar\chi}{\partial x^\lambda}\gamma^\lambda + \bar\chi\,V_\lambda{}^{(L)}\gamma^\lambda - \bar\chi\,P_\lambda{}^{(L)}\gamma^\lambda\gamma^5 = 0 \qquad (21\,\mathrm{b})$$

from which it follows that the conservation laws for $\bar\chi\,\gamma_\lambda\,\chi$ and $\bar\chi\,\gamma_\lambda\,\gamma_5\,\chi$ (conservation of electric and baryonic charge) do not hold unless $V_\lambda{}^{(L)}$ vanishes resp. is sufficiently small. An identical vanishing is not possible, except if we assume $\xi_{\lambda\mu,\nu} \equiv 0$ which would destroy also $P_\lambda{}^{(L)}$. But we can obtain an approximate validity of these conservation laws which means, as we shall see, that they hold exactly only locally, that is under strong interaction. For this purpose we impose on the $\xi_{\lambda\mu}$ the following condition

$$\xi_{\lambda\mu,\nu} + \xi_{\lambda\nu,\mu} \equiv 0\,. \qquad (22)$$

This can be satisfied if we assume for example

$$\xi_{\lambda\mu,\nu} = \pi_{[\lambda}\,\pi_{\mu,\nu]} \qquad (23)$$

which means a total antisymmetrization. Thus there remain four independent parameters π_λ according to the number of components of a total antisymmetric tensor of third rank (pseudovector). This leaves

$$P_\lambda{}^{(L)} = \pm\, l^2\,\bar\chi\,\gamma_\lambda\,\gamma_5\,\chi\,, \quad l = \text{const.} \qquad (24)$$

[4] Y. L. DALETSKIJ and S. G. KREIN. Dokl. Akad. Nauk SSSR **76**, 1 [1951].

[5] We recall that an affine connection is usually introduced by the concept of parallel displacement: The infinitesimal change which a vector undergoes if displaced along a curve $\tau = \tau(x)$ is given by $\delta q^\lambda = -C^\lambda{}_{\varrho\sigma}\, q^\varrho\,(\mathrm{d}x^\sigma/\mathrm{d}\tau)\ \mathrm{d}\tau$. If one requires that the norm of a vector should remain unaltered one obtains in the case of a euclidean metric the condition $C_{\lambda\mu\nu} = -C_{\mu\lambda\nu}$.

which gives the well known HEISENBERG–PAULI-term. Hence, $V_\lambda^{(L)}$ is already determined by (24). We proceed in demonstrating the approximate validity of the conservation laws for $\bar\chi\gamma_\lambda\chi$ and $\bar\chi\gamma_\lambda\gamma_5\chi$. Due to (22) we have

$$\xi^{\lambda\varrho}{}_{,\varrho}\equiv 0\,.\quad \xi_{\langle\lambda\mu,\nu\rangle}\equiv 3\,\xi_{\lambda\mu,\nu}\,. \qquad (25\,\text{a, b})$$

Now, since we have from (14)

$$a_{\lambda\mu}(\zeta)=\delta_{\lambda\mu}+\zeta\,\omega_{\lambda\mu}+O(\omega^2) \qquad (26)$$

it follows, expanding (13),

$$\gamma^\lambda\,\Omega^{-1}\frac{\partial\Omega}{\partial x^\lambda}=\frac{1}{4}\gamma^\lambda\,\gamma^\mu\,\gamma^\nu\,(\omega_{\alpha\beta,\lambda}-\omega^\sigma{}_{[\alpha}\,\omega_{\beta]\,\sigma,\lambda})+O(\omega^2)\,. \qquad (27)$$

From this we get with $\omega_{\lambda\mu}=\xi_{\lambda\mu}$ and (25 a, b)

$$V_\lambda^{(L)}=-\tfrac{1}{8}\,(\xi^{\varrho\sigma}\,\xi_{\sigma\lambda})_{,\varrho}+O(\xi^3)\sim O(\xi^2)\,, \qquad (28\,\text{a})$$

$$P_\lambda^{(L)}=\pm l^2\,\bar\chi\gamma_\lambda\gamma_5\chi=(-1)^\lambda\tfrac{1}{4}\,(3\,\xi_{\varrho\sigma,\tau}+\xi^\alpha{}_{\langle\varrho,\sigma}\xi_{\tau\rangle\alpha})+O(\xi^3)\sim O(\xi)\quad(\varrho\neq\sigma\neq\tau\neq\varrho;\ \lambda\neq\varrho,\sigma,\tau)\,. \qquad (28\,\text{b})$$

Hence we have from (21 a, b)

$$(\bar\chi\gamma^\varrho\chi)_{,\varrho}+O(\xi^2)=0\,,\quad(\bar\chi\gamma^\varrho\gamma^5\chi)_{,\varrho}+O(\xi^2)=0 \qquad (29\,\text{a, b})$$

which shows that the conservation laws hold in general only approximately up to terms of second order [however, we have $V^{\varrho(L)}_{,\varrho}+O(\xi^3)=0$]. The assertion we have made above, namely that the conservation laws hold exactly under strong interactions, will become clear from the following: As we have pointed out in the preceding chapter, we may choose our initial condition always in a way that at a fixed but arbitrary point $x=x_0$ we have $\xi_{\lambda\mu}(x_0)=0$ and hence

$$P_\lambda^{(L)}\big|_{x=x_0}=(-1)^\lambda\,\tfrac{3}{4}\,\xi_{\varrho\sigma,\tau}\big|_{x=x_0}=\pm l^2(\bar\chi\gamma_\lambda\gamma_5\chi)_{x=x_0}\,. \qquad (30)$$

To simplify the notation we shall occasionally write the pseudovector as a total antisymmetric tensor of third rank using the abbreviation $\gamma^\lambda\,\gamma^\mu\,\gamma^\nu=\gamma^{\lambda\mu\nu}$, $(\lambda\neq\mu\neq\nu\neq\lambda)$. From (30) we may conclude that the term of n-th order is of the magnitude of $[l^2(\bar\chi\gamma_\lambda\gamma_5\chi)]^n$. Neglecting terms of second and higher order we can use (30) as an approximation assuming that it holds at every point, i. e.

$$\xi_{\varrho\sigma,\tau}=\mp\tfrac{4}{3}\,l^2\,\bar\chi\,\gamma_{\varrho\sigma\tau}\,\chi+O(\xi^2)\,. \qquad (31)$$

From that it follows

$$\xi_{\varrho\sigma}=\mp\tfrac{4}{3}\,l^2(\bar\chi\,\gamma_{\varrho\sigma\tau}\,\chi)_{x=x_0}\,\Delta x^\tau+O(\xi^2)\,,\quad \Delta x^\tau=x^\tau-x_0{}^\tau, \qquad (32)$$

and with (28 a, b)

$$\gamma^\lambda\,\Omega^{-1}\frac{\partial\Omega}{\partial x^\lambda}=\pm\tfrac{1}{6}\,l^2\,\gamma^{\lambda\mu\nu}(\bar\chi\,\gamma_{\lambda\mu\nu}\,x)_{x=x_0}-\tfrac{2}{9}\,l^4[\gamma^\lambda(\bar\chi\,\gamma^{\varrho\sigma}{}_\tau\,\chi)\,(\bar\chi\,\gamma_{\sigma\lambda\varrho}\,\chi)$$
$$+\gamma^{\lambda\mu\nu}(\bar\chi\,\gamma^\sigma{}_{\mu\tau}\,\chi)\,(\bar\chi\,\gamma_{\lambda\sigma\nu}\,\chi)]_{x=x_0}\,\Delta x^\tau+O(\xi^3)\,. \qquad (33)$$

Since the point, we have chosen, is arbitrary, we would obtain the same expansion everywhere. This permits the following interpretation: Relation (33) represents an expansion in "interaction" terms with a decreasing strength; the strength of the term of n-th order is roughly given by $(l^2)^n$ whereas l^2 acts as a kind of fundamental coupling constant. The first term represents apparently a "strong" interaction. Since we have $V_\lambda^{(L)}(x_0)=0$ it follows that

$$(\bar\chi\gamma^\varrho\chi)_{,\varrho}\big|_{x=x_0}=0\,,\ (\bar\chi\gamma^\varrho\gamma^5\chi)_{,\varrho}\big|_{x=x_0}=0 \qquad (34\,\text{a, b})$$

which shows that the conservation laws are only exact "locally", i. e. under "strong" interactions; because at $x=x_0$ the other terms which represent consequently weaker interactions vanish as we see from (33). The expansion (33) makes it also plausible that the range of a certain type of interaction is, roughly speaking, the inverse of its strength, for the convergence of the expansion requires that (assuming $|\bar\chi\gamma_\lambda\gamma_5\chi|_{x=x_0}\sim 1$ for the sake of simplicity)

$$(l^2)^n\,|x-x_0|^n<1\,. \qquad (35)$$

Now, since the convergence is guaranted by the fact that the function, we have expanded is analytical [provided $(\bar\chi\gamma_\lambda\gamma_5\chi)_{x=x_0}$ is analytical], relation (35) is satisfied eo ipso and hence not a condition but a consequence.

III.

Let us now consider the special complex transformations Λ with $\omega_{\lambda\mu}=\eta_{\lambda\mu}$. We denote the (two-component) spinor here by φ and mark again the vector and pseudovector by an upper index: $V_\lambda^{(\Lambda)}\equiv V_\lambda(\Lambda)$, $P_\lambda^{(\Lambda)}\equiv P_\lambda(\Lambda)$. Due to the mixing up of real and imaginary components, we find that none of the components of $V_\lambda^{(\Lambda)}$ and $P_\lambda^{(\Lambda)}$ is pure real or imaginary.

Therefore we split up the vector and pseudovector according to

$$V_\lambda^{(A)} = V_\lambda^{(A\,\mathrm{I})} + V_\lambda^{(A\,\mathrm{II})}, \quad P_\lambda^{(A)} = P_\lambda^{(A\,\mathrm{I})} + P_\lambda^{(A\,\mathrm{II})} \tag{36 a, b}$$

with

$$V_k^{(A\,\mathrm{I,\,II})} = \tfrac{1}{2}\,(V_k^{(A)} \mp V_k^{*(A)}), \quad V_4^{(A\,\mathrm{I,\,II})} = \tfrac{1}{2}\,(V_4^{(A)} \pm V_4^{*(A)}), \tag{37 a}$$

$$P_k^{(A\,\mathrm{I,\,II})} = \tfrac{1}{2}\,(P_k^{(A)} \pm P_k^{*(A)}), \quad P_4^{(A\,\mathrm{I,\,II})} = \tfrac{1}{2}\,(P_4^{(A)} \mp P_4^{*(A)}). \tag{37 b}$$

Using (27) we find with (4) that

$$V_\lambda^{(A\,\mathrm{I})},\, P_\lambda^{(A\,\mathrm{I})} \sim O(\eta^2)\,; \quad V_\lambda^{(A\,\mathrm{II})},\, P_\lambda^{(A\,\mathrm{II})} \sim O(\eta)\,. \tag{38 a, b}$$

We demand again that $V_\lambda^{(A)}$ and $P_\lambda^{(A)}$ should be built up from the fundamental covariants of the field. This leads to

$$V_\lambda^{(A\,\mathrm{II})} \sim \bar\varphi\,\gamma_\lambda\,\varphi \quad \text{and} \quad P_\lambda^{(A\,\mathrm{II})} \sim i\,\bar\varphi\,\gamma_\lambda\,\gamma_5\,\varphi\,.$$

But the second relation must be excluded for reasons which are quite analogous to those already given in the preceding chapter: The conservation laws for $\bar\varphi\,\gamma_\lambda\,\varphi$ and $\bar\varphi\,\gamma_\lambda\,\gamma_5\,\varphi$ do not hold unless $V_\lambda^{(A\,\mathrm{I})}$ and $P_\lambda^{(A)}$ vanish (resp. are sufficiently small). Again we can obtain by a treatment, which is similiar to the one given above, that the conservation laws hold approximately, i. e. they are locally exact. To obtain this we must reduce $P_\lambda^{(A)}$ to a second order term by imposing a symmetry condition. This is achieved by

$$\eta_{\lambda\mu} = \tfrac{1}{2}\,(\vartheta_{\lambda,\,\mu} - \vartheta_{\mu,\,\lambda}) \equiv \vartheta_{[\lambda,\,\mu]} \equiv 3\,\xi_{\lambda\mu,\,\nu} \tag{39}$$

which leaves four independent parameters. From (39) it follows that

$$\eta_{\langle\lambda\mu,\nu\rangle} \equiv \eta_{\lambda\mu,\,\nu} + \eta_{\mu\nu,\,\lambda} + \eta_{\nu\lambda,\,\mu} \equiv 0 \quad (\lambda \neq \mu \neq \nu \neq \lambda). \tag{40}$$

Hence, if we assume

$$V_\lambda^{(A\,\mathrm{II})} = \pm\,k^2\,\bar\varphi\,\gamma_\lambda\,\varphi \tag{41}$$

then $P_\lambda^{(A)}$ and $V_\lambda^{(A\,\mathrm{I})}$ are already determined by $V_\lambda^{(A\,\mathrm{II})}$. Thus we get

$$V_\lambda^{(A\,\mathrm{II})} = \pm\,k^2(\bar\varphi\,\gamma_\lambda\,\varphi) = \tfrac{1}{4}\,\eta_\lambda{}^{\varrho}{}_{,\,\varrho} + O(\eta^2), \tag{42 a}$$

$$V_\lambda^{(A\,\mathrm{I})},\; P_\lambda^{(A\,\mathrm{I,\,II})} \sim O(\eta^2) \tag{42 b}$$

and the conservation laws hold approximately, i. e. we have

$$(\bar\varphi\,\gamma^\varrho\,\varphi)_{,\,\varrho} + O(\eta^2) = 0\,, \quad (\bar\varphi\,\gamma^\varrho\,\gamma^5\,\varphi)_{,\,\varrho} + O(\eta^2) = 0\,. \tag{43 a, b}$$

Since we may again at an arbitrary but fixed point $x = x_0$ choose our initial condition so that we have $\eta_{\lambda\mu}(x_0) = 0$, we get with

$$(\bar\varphi\,\gamma^\varrho\,\varphi)_{,\,\varrho}\big|_{x=x_0} = 0, \quad (\bar\varphi\,\gamma^\varrho\,\gamma^5\,\varphi)_{,\,\varrho}\big|_{x=x_0} = 0 \tag{44 a, b}$$

again a strict local conservation. We note that the relations (39) and (42 a) (as well as the fact that $i\,\bar\varphi\,\gamma_\lambda\,\varphi$ can be related to the current vector) suggest the identification of $\frac{1}{8}\,\vartheta_\lambda$ with $i\,A_\lambda$ where A_λ is the electromagnetic field. This would lead to a certain type of nonlinear electrodynamics by taking into consideration terms of second and higher order.

We have found that the real transformations can be characterized by a pseudovector and the special complex transformations by a vector coupling. It can be shown (and we shall give an argument below) that the pseudovector coupling has a greater symmetry than the vector coupling. This leads to the assumption that the real local transformations represent possibly the baryonic and consequently the special complex transformations the leptonic interactions. The latter assumption is enforced by the circumstance that the parameters of the local complex special transformations can be related to the electromagnetic field in a way we have shown above. Hence, (34 a, b) would describe the conservation of electric and baryonic charge and (44 a, b) the conservation of electric and leptonic charge. Now, since the group of general complex orthogonal transformations (which consists of both real and special complex transformations and hence would describe the interaction between baryons and leptons) requires a four-component spinor, the question rises whether this is not equivalent to the introduction of isospin. Such an equivalence would apparently correspond to the assumption that one of the isospin componenst is replaced by a bounded state consisting of a baryon (of the other isospin component) and leptons. In the case of the neutron, for example, this would lead to the components of the β-decay. Of course, one can introduce the isospin also by doubling the components of χ using a representation $\gamma \sim \sigma \times \tau$ in which the γ-matrices are the direct products of the matrices of isospin and spin [6]. But the interpretation above would have the advantage that it permits a distinction between "elementary" and "composed" baryons, thus making for example the instability of the neutron resp. the β-decay plausible.

[6] See H. P. Dürr, Z. Naturforschg. **16 a**, 327 [1961].

IV.

If we permit both real and special complex transformations, we have, with regard to the interpretation given above, an interaction between baryons and leptons. In order to recover the equations we have obained in the special cases, we assume the following general equation:

$$(\gamma^\lambda)\,\frac{\partial\psi}{\partial x^\lambda}+(\gamma^\lambda)\,V_\lambda\,\psi+(\gamma^\lambda)\,(\gamma^5)\,P_\lambda\,\psi=0 \qquad (45)$$

with

$$\psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix},\quad (\gamma^\lambda)=\begin{pmatrix}\gamma^\lambda & 0\\ 0 & \gamma^\lambda\end{pmatrix},\quad (\gamma^5)=\begin{pmatrix}\gamma^5 & 0\\ 0 & \gamma^5\end{pmatrix} \qquad (46\,\text{a, b, c})$$

whereas φ describes the lepton and χ the baryon states. We recall that ψ is in fact a four-component spinor (φ and χ are both two-component spinors). However, we use for φ and χ a four- and consequently for ψ an eight-component notation. V_λ and P_λ are given by

$$V_\lambda=\tfrac{1}{4}\,\omega_{\alpha\beta,\varrho}\int_0^1 a^\alpha{}_\lambda(\zeta)\;a^\beta{}_\varrho(\zeta)\;\mathrm{d}\zeta\,, \qquad (47\,\text{a})$$

$$P_\lambda=\tfrac{1}{4}\,(-1)^\lambda\,\omega_{\alpha\beta,\langle\varrho}\int_0^1 a^\alpha{}_\sigma(\zeta)\;a^\beta{}_{\tau\rangle}\;(\zeta)\;\mathrm{d}\zeta\,, \qquad (\lambda\neq\varrho,\sigma,\tau) \qquad (47\,\text{b})$$

and the coupling is expressed by

$$V_\lambda^{(\mathrm{II})}=\pm k^2\,\bar\varphi\,\gamma_\lambda\,\varphi\,,\quad P_\lambda^{(\mathrm{II})}=\pm l^2\,\bar\chi\,\gamma_\lambda\,\gamma_5\,\chi\,. \qquad (48\,\text{a, b})$$

We recall that by definition

$$V_k^{(\mathrm{II})}=\tfrac{1}{2}\,(V_k-V_k^*),\quad V_4^{(\mathrm{II})}=\tfrac{1}{2}\,(V_4+V_4^*), \qquad (49\,\text{a})$$

$$P_k^{(\mathrm{I})}=\tfrac{1}{2}\,(P_k-P_k^*),\quad P_4^{(\mathrm{I})}=\tfrac{1}{2}\,(P_4+P_4^*) \qquad (k=1,2,3) \qquad (49\,\text{b})$$

and that further

$$\omega_{\lambda\mu}=\xi_{\lambda\mu}+\eta_{\lambda\mu};\quad \xi_{kl},\,\eta_{4l}\ \text{real};\ \eta_{kl},\,\xi_{4l}\ \text{imaginary} \qquad (k,l=1,2,3) \qquad (50)$$

with

$$\xi_{\lambda\mu,\nu}=\pi_{[\lambda}\,\pi_{\mu,\nu]}\,,\quad \eta_{\lambda\mu}=\vartheta_{[\lambda,\mu]}\,. \qquad (51\,\text{a, b})$$

Hence, V_λ and P_λ are determined by φ and χ.

Neglecting terms of second and higher order and taking into consideration the relations (47) – (51) we get from (45) the following system of equations:

$$\gamma^\lambda\;\pm k^2\,\gamma^\lambda\,\varphi\,(\bar\varphi\,\gamma_\lambda\,\varphi)\;\pm l^2\,\gamma^\lambda\,\gamma^5\,\varphi\,(\bar\chi\,\gamma_\lambda\,\gamma_5\,\chi)\approx 0\,, \qquad (52\,\text{a})$$

$$\gamma^\lambda\,\frac{\partial\chi}{\partial x^\lambda}\pm k^2\,\gamma^\lambda\,\chi\,(\bar\varphi\,\gamma_\lambda\,\varphi)\;\pm l^2\,\gamma^\lambda\,\gamma^5\,\chi\,(\bar\chi\,\gamma_\lambda\,\gamma_5\,\chi)\approx 0\,. \qquad (52\,\text{b})$$

Within this approximation we obtain from (52 a, b) the following conservation laws

$$(\bar\varphi\,\gamma^\varrho\,\varphi)_{,\varrho}\approx 0\,,\quad (\bar\varphi\,\gamma^\varrho\,\gamma^5\,\varphi)_{,\varrho}\approx 0\,, \qquad (53\,\text{a, b})$$

$$(\bar\chi\,\gamma^\varrho\,\chi)_{,\varrho}\approx 0\,,\quad (\bar\chi\,\gamma^\varrho\,\gamma^5\,\chi)_{,\varrho}\approx 0\,. \qquad (54\,\text{a, b})$$

With regard to the interpretation given above, we identify them with the conservation laws for the electric, leptonic and baryonic charge respectively. It is clear from the foregoing that these conservation laws do not longer hold if we take into consideration the terms with higher order, i. e. weaker interactions.

We conclude with a remark concerning a possible interpretation of the electromagnetic field. Within the approximation we have used in (52 a, b), we may subject the parameters ϑ_λ to a gauge condition which corresponds to (53):

$$\vartheta^\varrho{}_{,\varrho}\approx 0\,. \qquad (55)$$

Recalling that we have set in agreement with the reality conditions $\frac{1}{8}\,\vartheta_\lambda=i\,A_\lambda$ where A_k real, A_4 imaginary, we have up to terms of higher order

$$\mp i\,k^2\,\bar\varphi\,\gamma_\lambda\,\varphi=\Box\,A_\lambda+O(A^2),\ \left(\Box\equiv\frac{\partial}{\partial x^\varrho}\,\frac{\partial}{\partial x_\varrho}\right) \qquad (56)$$

which justifies an identification of A_λ with the electromagnetic field. If we take into consideration the terms of higher order, we get apparently a type of nonlinear electrodynamics.

Conclusion

The foregoing considerations give a very rough sketch and there remains a great number of questions among which first of all are those which ask for the behaviour under symmetry operations. Furthermore, one has to investigate in detail whether the isospin can be introduced in the way we have proposed above. We note here that in the case $V_\lambda\sim O(\omega^2)$, we have two additional approximate conservation laws, namely for $\psi^{\mathrm{T}}\,C\,\gamma_\lambda\,\psi$ and $\psi^{\mathrm{T}}\,C\,\gamma_\lambda\,\gamma_5\,\psi$, which are possibly related to the conservation of isospin. This proves on the other hand a greater symmetry of the pseudovector coupling since such a coupling obviously does not destroy the mentioned conservation laws. We recall here that the first order vector term is due to the φ-states while the χ-states produce only second order vector terms. This underlines the assumed connection between the real local orthogonal transformations and the baryon interaction on the one hand and the special complex local orthogonal transformations and the lepton interactions on the other hand. We note further that the quantities $\psi^{T} C \gamma_{\lambda} \psi$ and $\psi^{T} C \gamma_{\lambda} \gamma_{5} \psi$ are covariants not only under real but also under special complex transformation as can be proved easily by using relation (8 c). As to the quantization, we have already remarked above, that one can perhaps make use of the linearity of the field equation in the "intrinsic" system by writing down the commutatorfunction in that system and then changing over to the "physical" system by means of local transformations.

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Performance of Thermal Diffusion Columns for Gas Mixtures

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A systematic study of the performance of thermal diffusion columns is carried out in order to examine the limits of validity of the theory of isotopic mixtures in case of mixtures of gases. Nitrogen-carbon dioxide mixtures are used. It is found that the pressure dependence of the separation factor has the same qualitative form as that derived from the simplified theory valid for isotopic mixtures.

The theoretical hydrodynamics of the performance of the CLUSIUS–DICKEL thermal diffusion column has been examined by CLUSIUS and DICKEL [1], JENSEN [2], and WALDMANN [3] of the German School; by FURRY, JONES, and ONSAGER [4], JONES and FURRY [5], and McINTEER and REISFELD [6] of the American School; and by SRIVASTAVA and SRIVASTAVA [7], and SAXENA and RAMAN [8] of the Indian School.

The only well established theory known until now is based on some simplifying conditions [8], the most important of which is that the dependence of the coefficient of viscosity (η), thermal conductivity (λ), diffusion (D), density (ϱ), and thermal diffusion factor (α_T) on the composition of the gas mixture is negligible; only temperature dependence of these quantities is taken into consideration. In other words, the only important variation of these quantities with position is that due to the existence of temperature gradients. This condition is justified only in case of isotopic mixtures where the fractional difference of the molecular weight is small. Another important restriction imposed on the theory is that, in limited temperature ranges, the inverse power model is capable of effectively representing the interaction among molecules for the calculation of η and D.

Under these simplifying conditions, it can be shown that for closed column operation at the steady state

$$\ln q = H L / (K_c + K_d) \tag{1}$$

where q is the equilibrium-separation factor and is the ratio of (x_1/x_2) at the top end of the column to that at the bottom end, x_i being the molar concentration of species i; L is the length of the column. The theoretical expressions for H, K_c, and K_d (which are functions of the transport coefficients of the gas mixture as well as the geometry of the column) indicate that these quantities are proportional to p^2, p^4, and p^0, respectively. Therefore, equation (1) can be put in the form

$$\ln q = \frac{a/p^2}{1+b/p^4} \tag{2}$$

where a and b are given by

$$a = H L p^2/K_c, \qquad b = K_d p^4/K_c \tag{3}$$

and p is the pressure of the gas.

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